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THE VEHICLE ROUTING PROBLEM WITH STOCHASTIC TRAVEL TIMES

by

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ABSTRACT

This paper considers vehicle routing problems (VRPs) with stochastic service and travel times, in which vehicles incur a penalty proportional to the duration of their route in excess of a preset constant. Three mathematical programming models are described: a chance constrained model, a two-index recourse model and a three-index recourse model. It is shown how a particular class of stochastic VRPs reduce to the deterministic case. A general branch-and-bound algorithm for the three models is described. Computational results indicate that moderate size problems can be solved to optimality.

Key-words : stochastic vehicle routing problems, chance constrained models, recourse models.

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RÉSUMÉ

On considère dans cet article des problèmes de tournées avec temps de service et temps de voyage stochastiques, dans lesquels les tournées en excès d'une certaine valeur engendrent une pénalité. Trois modèles de programmation mathématique sont présentés : un modèle avec contraintes probabilistes, un modèle avec recours à deux indices ainsi qu'un modèle avec recours à trois indices. On montre comment dans certains cas, le problème peut se réduire au cas déterministe. On procède ensuite à la description d'un algorithme de "branch and bound" applicable aux trois modèles. Les résultats numériques indiquent que l'approche proposée permet de résoudre de façon exacte des problèmes de taille moyenne.

Mots-clés : problèmes de tournées stochastiques, modèles probabilistes, modèles avec recours.

INTRODUCTION

The **vehicle routing problem** (VRP) can be defined as follows. Let $G=(V,E)$ be a graph where $V = \{v_0, v_1, \ldots, v_n\}$ is the **vertex set** and $E = \{(v_i, v_j): v_i, v_j \in V, i \neq j\}$ is the **arc set.** Vertex v_0 represents a depot at which are based a maximum of m vehicles, and $v_1, \ldots, v_n$ correspond to cities or customers. With E is associated a **distance** or **travel cost** matrix $C=(c_{ij})$. If C is symmetrical, as will be the case in this paper, E can be defined as a set of undirected **edges** (v_i, v_j) where $i<j$. The VRP consists of determining a set of at most m vehicle routes

(i) starting and ending at the depot and such that

(ii) every vertex of $V \backslash \{v_0\}$ is visited exactly once by exactly one vehicle;

(iii) some side constraints on the vehicle routes are satisfied;

(iv) a linear combination of the number of vehicles and of the total routing cost is minimized.

Note that when C satisfies the triangle inequality (i.e. $c_{ij} \leq c_{ik} + c_{kj}$ for all i,j,k), condition (ii) is not restrictive. This occurs, for example, if c_{ij} is the length of a shortest path between i and j. This definition of the VRP also includes the case where the number of vehicles is fixed at a preset value m. There exists a wide variety of side constraints; for examples and recent surveys on the VRP, see Laporte and Nobert [10] and Golden and Assad [5].

This statement of the VRP does not adequately describe a variety of practical situations where one or several parameters are **stochastic.** This is the case, for example, where vehicles of fixed capacities must collect random quantities at the various customers (see for example Laporte, Louveaux and Mercure [8] as well as Dror, Laporte and Trudeau

[4]). Another case is where customers are visited on a random basis (Jaillet [6] and Bertsimas [1]). In this paper, we consider a particular class of stochastic VRP's: the **vehicle routing problem with stochastic travel times** (SVRP). With E is associated a stochastic **travel time** matrix $T=(t_{ij})$. Similarly, a stochastic **service time** vector $\tau=(\tau_i)$ is defined on $V\backslash\{v_0\}$. Vehicles incur a penalty proportional to the duration of their route in excess of a preset constant B. Such problems occur frequently in contexts where drivers must be paid overtime for work done after normal hours. A variant of this problem is where the penalty for route duration on excess of B is proportional to the vehicle content at arrival. Such is the case when armoured vehicles are used to collect money from bank branches to a central bank: all transactions registered after a certain time of day are credited the next day, thus losing one day's interest. This variant can be treated the same way as the stochastic **vehicle routing problem** VRP with uncertain supplies.

Stochastic VRPs are considerably more difficult to solve than their deterministic counterparts and have thus received less attention from operations researchers. To the authors' knowledge, this problem has never been previously treated in the Operational Research literature. The only related study appears to be Leipälä's paper [11] which considers the problem of estimating the length of the optimal tour in a traveling salesman problem with stochastic distances. Our objective is to provide workable formulations and exact algorithms for SVRPs of moderate sizes.

This paper is organized as follows. In section 1, the SVRP is formulated as a chance constrained problem. Then follow in sections 2 and 3 two recourse models for the same problem. It is shown in section 4 how a family of SVRPs reduces to the deterministic case. A general algorithm for the models of sections 1, 2 and 3 is described in section 5 while computational results are presented in section 6. The conclusion follows.

1. CHANCE CONSTRAINED MODEL

The objective in the chance constrained model is to minimize a linear combination of vehicle and routing costs while ensuring that the probability of the duration of a route exceeding B is at most equal to a given value α. Define

x_{ij} $(i<j)$: the number of times edge (v_i,v_j) is used in the optimal solution. This variable is equal to 0 or 1 if $0<i<j$ and to 0, 1 or 2 if $i=0$; the case $x_{0j}=2$ corresponds to a return trip between the depot and v_j;

f : the fixed cost (per planning period) of a vehicle.

The model is then

$$(P1) \qquad \min \quad fm + \sum_{i<j} c_{ij}\, x_{ij}$$

subject to

$$\sum_{j=1}^{n} x_{0j} = 2m \tag{1}$$

$$\sum_{i<k} x_{ij} + \sum_{j>k} x_{kj} = 2 \qquad (v_k \in V\setminus\{v_0\}) \tag{2}$$

Illegal route elimination constraints (3)

$$x_{ij} \in \{0,1\} \qquad (v_i, v_j \in V\setminus\{v_0\}) \tag{4}$$

$$x_{0j} \in \{0,1,2\} \qquad (v_j \in V\setminus\{v_0\}) \tag{5}$$

All constraints except (3) are self-explanatory. Illegal routes referred to in constraints (3) include subtours disconnected from the depot and vehicle routes whose probability of having a duration in excess of B is larger than α. As in the travelling salesman problem (TSP) (Dantzig,

Fulkerson and Johnson [3]) illegal subtours can be eliminated by imposing the connectivity constraints

$$\sum_{i\in S, j\notin S \text{ or } i\notin S, j\in S} x_{ij} \geq 2 \qquad (S\subseteq V\setminus\{v_0\},\ 3\leq|S|\leq n-3)\ . \qquad (6)$$

Now consider a vehicle route $L = (v_0, v_{i_1},\ldots,v_{i_r},v_0)$. Such a route is illegal if

$$P\Big(\sum_{k=0}^{r}\big[t_{i_k i_{k+1}} + \tau_{i_k}\big] > B\Big) > \alpha\ , \qquad (7)$$

where $i_{r+1} = i_0$ and $\tau_{i_0} = 0$. In this case, L can be eliminated by imposing

$$\sum_{k=0}^{r} x_{i_k i_{k+1}} \leq r\ . \qquad (8)$$

Note that this constraint only eliminates route L, but none of the routes obtained by permuting the components of L. A stronger constraint can be derived under the following assumption. Let $S\subseteq V\setminus\{v_0\}$ and L_S be a permutation of the elements of S corresponding to the cities of a route on S starting and ending at the depot. Let also $t(L_S)$ be the total duration of L_S (including service time), $c(L_S)$ its total cost, and $L_S^* = \text{argmin}\ \{c(L_S)\}$ be the least cost permutation of the cities of S.

Assumption 1

$$P(t(L_S) > s) \geq P(t(L_S^*) > s) \text{ for all } s\geq B\ .$$

Note that this assumption is weaker than the assumption of monotonicity of t with respect to c, i.e.

$$c(L_S) > c(L_S^*) \Rightarrow t(L_S) > t(L_S^*) \text{ a.s.}$$

Also observe that if travel times are deterministic and service times are the only source of uncertainty, then Assumption 1 reduces to $t(L_S) > t(L_S^*)$.

Proposition 1

If L_S^* satisfies (7) and Assumption 1 holds, then the following constraint is valid:

$$\sum_{\substack{v_i \in S, v_j \notin S \\ \text{or } v_i \notin S, v_j \in S}} x_{ij} \geq 2\, V_\alpha(S) \qquad (S \subseteq V \setminus \{v_0\}), \tag{9}$$

where $V_\alpha(S)$ is a lower bound on the number of vehicles required to visit v_0 and all cities of S. It can be taken as the smallest integer satisfying

$$P(t(L_S^*) > B\, V_\alpha(S)) \leq \alpha . \tag{10}$$

Proof. By Assumption 1, if L_S^* satisfies (10), there is no other permutation L_S such that (10) would be satisfied with a strictly smaller value of $V_\alpha(S)$. Hence, at least $V_\alpha(S)$ vehicles are required to visit v_0 and all cities of S, and twice as many edges must link S and its complement. This is expressed by contraints (9). □

This chance constrained model is attractive and relatively simple to solve. However, in order to take into account the expected cost related to excess route duration, it is necessary to resort to a recourse model. Two such models are presented in the following two sections.

2. A THREE-INDEX RECOURSE MODEL

Recourse models for the SVRP are defined as follows. In the first stage, decisions are made about the number of vehicles and their routes. In the second stage, penalties are incurred for excess duration. We now describe a three-index recourse model. Define

$x_{ij\ell}$ $(i<j)$: the number of times (0, 1 or 2) edge (v_i, v_j) is traversed by vehicle ℓ in the optimal solution;

$$z_{k\ell} = \begin{cases} 1 & \text{if } v_k \text{ is visited by vehicle } \ell; \\ 0 & \text{otherwise;} \end{cases}$$

y_ℓ^ξ : the excess duration of route ℓ in state of the world ξ;

$c_{ij\ell}$: the travel cost of vehicle ℓ on edge (v_i, v_j);

$t_{ij\ell}^\xi$: the travel time of vehicle ℓ on edge (v_i, v_j) in state of the world ξ;

$\tau_{i\ell}^\xi$: the service time of vertex v_i by vehicle ℓ;

γ_ℓ : the (positive) unit penalty cost for excess duration on route ℓ;

B_ℓ : the value over which penalty is incurred on route ℓ.

The model is then

$$(P2) \quad \min f \sum_{\ell=1}^{m} z_{0\ell} + \sum_{\ell=1}^{m} \sum_{i<j} c_{ij\ell}\, x_{ij\ell} + E_\xi \Big(\sum_{\ell=1}^{m} \gamma_\ell\, y_\ell^\xi \Big)$$

subject to:

$$\sum_{\ell=1}^{m} z_{k\ell} = 1 \qquad (v_k \in V \setminus \{v_0\}) \qquad (11)$$

$$\sum_{j=1}^{n} x_{0j\ell} = 2\, z_{0\ell} \qquad (\ell=1,\ldots,m) \qquad (12)$$

$$\sum_{i<k} x_{ik\ell} + \sum_{j>k} x_{kj\ell} = 2\, z_{k\ell} \qquad (v_k \in V\setminus\{v_0\};\ \ell=1,\ldots,m) \qquad (13)$$

$$\sum_{\substack{i,j\in S \\ i<j}} x_{ij\ell} \leq |S| - 1 \qquad (S\subseteq V\setminus\{v_0\},\ 3\leq|S|\leq n-3;\ \ell=1,\ldots,m) \qquad (14)$$

$$y_\ell^\xi \geq \sum_{i<j} t_{ij\ell}^\xi\, x_{ij\ell} + \frac{1}{2}\sum_{i<j} (\tau_{i\ell}^\xi + \tau_{j\ell}^\xi)\, x_{ij\ell} - B_\ell \qquad (\ell=1,\ldots,m;\ \xi\in\Xi) \qquad (15)$$

$$x_{ij\ell} \in \{0,1\} \qquad (v_i, v_j \in V\setminus\{v_0\};\ \ell=1,\ldots,m) \qquad (16)$$

$$x_{0j\ell} \in \{0,1,2\} \qquad (v_j \in V\setminus\{v_0\};\ \ell=1,\ldots,m) \qquad (17)$$

$$z_{k\ell} \in \{0,1\} \qquad (v_k \in V;\ \ell=1,\ldots,m) \qquad (18)$$

$$y_\ell^\xi \geq 0 \qquad (\ell=1,\ldots,m;\ \xi\in\Xi) \qquad (19)$$

Constraints (11)-(14) and (16)-(19) of this model are standard or self-explanatory. Constraints (15) combined with (19) imply that a penalty will be incurred by vehicle ℓ only if the duration of its route, including service time, exceeds B_ℓ.

Note that the second stage problem is a simple recourse model with continuous decision variables. It follows that the second-stage expected optimal value function is a convex function of $x=(x_{ij\ell})$ and that it is piecewise linear in x if Ξ has a finite support.

3. A TWO-INDEX RECOURSE MODEL

We now present a two-index model for the SVRP. Here, variables x_{ij} indicate the absence or the presence of a trip on (v_i,v_j), without identifying which vehicle makes the trip. This obviously results in an important saving in the number of first stage variables. However, particular vehicle characteristics can no longer be taken into account so that it is now necessary to assume that all vehicles are identical. In addition to the variables already introduced in the previous two models, define

y : the total excess duration of a solution;

γ : the (positive) unit cost of excess duration;

$S_r = \{(v_i,v_j) : v_i,v_j \in V\setminus\{v_0\}$ and (v_i,v_j) belongs to the r^{th} feasible SVRP solution$\}$ (Note that $|S_r| = n-m$ for all r.);

Θ_r : the total expected excess duration of solution r.

The model is then:

$$(P3) \qquad \min \quad fm + \sum_{i<j} c_{ij}x_{ij} + \gamma y$$

subject to:

(1)-(5) and

$$y \geq \Theta_r \sum_{(v_i,v_j)\in S_r} x_{ij} - \Theta_r(n-m-1) \qquad \text{(all } r\text{)} \qquad (20)$$

$$y \geq 0 . \qquad (21)$$

In this model, constraints (3) are defined as in (6). Constraints (1)-(5) ensure that the solution is feasible for the m-TSP (Laporte and Nobert [9]), where m is either fixed or variable. Constraints (20) and (21), with the fact that y appears with a positive coefficient in the

objective function ensure that a penalty equal to θ_r will be incurred only if solution r is selected, since $\sum_{(v_i,v_j)\in S_r} x_{ij}$ is at most equal to n-m and equal to that value only if solution r is selected.

4. REDUCTION OF THE RECOURSE MODEL TO THE DETERMINISTIC CASE

In this section, we show that, under certain conditions, the recourse model (P2 or P3) reduces to a deterministic problem. This occurs, for example, under the following scenario. Suppose all τ_i's are deterministic and all $t_{ij\ell}$'s simultaneously assume one of two values $t^-_{ij\ell}$ or $t^+_{ij\ell} = \lambda t^-_{ij\ell}$, $\lambda>1$, with probabilities p^- and $p^+ = 1-p^-$, respectively. Such is the case of travel times on a road network under two main congestion levels. In such a context, it is natural to also impose bounds on the minimal duration of a route. The formulation of the recourse model is then

$$(P4)\quad \min \quad f \sum_{\ell=1}^{m} z_{0\ell} + \sum_{\ell=1}^{m} \sum_{i<j} c_{ij\ell} x_{ij\ell} + E_\xi\Big(\sum_{\ell=1}^{m} \gamma_\ell y^\xi_\ell\Big)$$

subject to

(11)-(19), and

$$\beta_\ell B_\ell \le \sum_{i<j} t^-_{ij\ell} x_{ij\ell} + \frac{1}{2} \sum_{i<j} (\tau_{i\ell}+\tau_{j\ell})\, x_{ij\ell} \le B_\ell \,, \qquad (22)$$

where $0<\beta_\ell<1$.

Constraint (22) ensures that in the best circumstances, planned route durations do not exceed B_ℓ and have a "productivity factor" at least equal to β_ℓ.

Now define $S_\ell = \frac{1}{2} \sum_{i<j} (\tau_{i\ell}+\tau_{j\ell})\, x_{ij\ell}$ to be the service time on route ℓ and let $\rho_\ell = S_\ell/B_\ell$ for $\ell=1, \ldots, m$.

Proposition 2

If $\beta_\ell > \rho_\ell + (1-\rho_\ell)/\lambda$ for $\ell=1, \ldots, m$, then (P4) is either infeasible or deterministic.

Proof. If $\rho \geq 1$ for some ℓ, then (P4) is infeasible. Now consider $\rho_\ell < 1$ for $\ell=1, \ldots, m$. Suppose (P4) is feasible and define $T_\ell = \sum_{i<j} t^-_{ij\ell} x_{ij\ell}$. By (22), $T_\ell \geq \beta_\ell B_\ell - S_\ell = \left(\frac{\beta_\ell B_\ell - S_\ell}{B_\ell - S_\ell} \right)(B_\ell - S_\ell)$. By assumption, $\beta_\ell > \rho_\ell + (1-\rho_\ell)/\lambda$, which implies $\frac{\beta_\ell B_\ell - S_\ell}{B_\ell - S_\ell} > \frac{1}{\lambda}$ and $\lambda T_\ell + S_\ell > B_\ell$ for all ℓ. Therefore, for all λ, constraint (15) is binding when $t=(t^+_{ij\ell})$ occurs and non-binding when $t=(t^-_{ij\ell})$. In the objective function of (P4), $E_\xi(\sum_{\ell=1}^{m} \gamma_\ell y^\xi_\ell)$ reduces to $p^+ \sum_{\ell=1}^{m} (\lambda \sum_{i<j} t^-_{ij\ell} x_{ij\ell} + \frac{1}{2} \sum_{i<j} (\tau_{i\ell} + \tau_{j\ell}) x_{ij\ell} - B_\ell)$, i.e. (P4) is deterministic. □

5. ALGORITHM

We briefly describe in this section a general algorithm for the three models (P1), (P2) and (P3). Then follow a number of comments specific to each particular model.

The constraints contained in these three models can be divided into five categories:

(i) **degree constraints** which specify the vehicles used at various vertices (11) and the degree of each vertex [(1),(2),(12),(13)];

(ii) **lower and upper bounds** on the variables;

(iii) **integrality requirements** on the variables;

(iv) **illegal route and subtour elimination constraints** [(6),(8) or (9),(14)].

(v) **lower bounds on penalties** for excess route durations [(15),(19), (20),(21)].

The general branch and bound algorithm for the three models can be outlined as follows:

Step 0. Define the current problem as the relaxed problem containing constraints (i) and (ii).

Step 1. Solve the current problem.

Step 2. Check for violated integrality conditions. If the solution is not integer, gradually regain integrality by branching on fractional variables

Step 3. At an integer solution, check first for subtours and then for illegal routes. If any violation is detected, create a new current problem by generating appropriate route or subtour elimination constraints and go to step 1.

Step 4. At a feasible solution, compute the exact expected penalty for excess duration. If it exceeds the higher lower bound provided by the current problem, create a new current problem by introducing the appropriate type (v) constraint and go to step 1. Otherwise update the incumbent. □

The algorithm terminates when all nodes of the search tree have been explored according to the usual branch and bound rules. Convergence is guaranteed by this process and by the fact that the number of (potential) constraints in each model is finite.

In the chance constrained model (P1), constraints (v) and hence step 4 do not apply. Otherwise, the algorithm is identical to that of the deterministic distance constrained VRP (Laporte, Desrochers and Nobert [7]).

When Assumption 1 is satisfied, some difficulties may arise in the computation of the value of L_S^* required for the determination of $V_\alpha(S)$, since it involves solving a TSP on $SU\{v_0\}$. An easy case occurs when C satisfies the triangle inequality.

Proposition 3

Assume C satisfies the triangle inequality and consider an optimal solution to the current problem containing no subtours identified in step 3 of the algorithm. Let L_S be the route corresponding to this optimal solution. Then $L_S^* = L_S$.

Proof. The current solution is feasible and optimal for the m-TSP at the current node of the search tree. Since C satisfies the triangle inequality, each vehicle route L_S taken separately is also optimal: (i) permuting or partitioning it cannot decrease the total distance required to serve $SU\{v_0\}$, (ii) moreover, sharing its vertices between several routes including vertices in $V\backslash(SU\{v_0\})$ cannot yield a shorter distance since for any vertex sequence $(v_{i_1}, v_{i_2}, v_{i_3})$ on a route L with $v_{i_2} \notin S$, a route of equal or shorter distance can be obtained by deleting v_{i_2} and connecting v_{i_1} to v_{i_3}. Hence, L_S^* is the current route on $SU\{v_0\}$. □

Note that constraints (9) generated with this value of L_S^* must be removed when backtraking, since the length of L_S^* is affected by the fixed x_{ij} variables at the current node of the search tree.

In the three-index recourse model (P2), whenever a type (v) constraint is generated for a given vehicle ℓ, it is also generated for all other vehicles in order to avoid the occurrence of an equivalent solution in which route indices are merely permuted. In this respect, the algorithm is similar to the multi-cut approach to outer-approximation in two-stage stochastic programming (Birge and Louveaux [2]).

Finally, in the two-index recourse model (P3), constraints (20) provide a non-trivial lower bound on y only when the current first stage solution coincides with a first stage solution previously obtained. In order to gain more information on y, it is useful to introduce in the first problem considered at step 0, a more accurate lower bound on y. Such a bound is now provided.

Proposition 4

A valid lower bound on y is given by

$$y \geq E_\xi \Big(\sum_{i<j} t_{ij}^\xi x_{ij} + \frac{1}{2} \sum_{i<j} (\tau_i^\xi + \tau_j^\xi)\, x_{ij} - mB \Big)^+ , \qquad (23)$$

where $(x)^+ = \max(x,0)$.

Proof. The validity of this bound rests on the fact that for any given ξ, the expression within parentheses compares an aggregate of all travel and service times with an aggregate of available time on each route. □

In practice, incorporating (23) into a linear program poses no difficulty when the number of states ξ is finite. Let the states be indexed by $k=1,\ldots,K$ and p^k represent their respective probabilities. Then (23) is equivalent to

$$y \geq \sum_{k=1}^{K} p^k\, y^k \qquad (24)$$

$$y^k \geq \sum_{i<j} t_{ij}^k\, x_{ij} + \frac{1}{2} \sum_{i<j} (\tau_i^k + \tau_j^k)\, x_{ij} - mB \qquad (k=1,\ldots,K) \quad (25)$$

and

$$y^k \geq 0 \qquad (k=1,\ldots,K). \quad (26)$$

□

These constraints were used in the implementation of our algorithm for (P3).

6. COMPUTATIONAL RESULTS

Of the three models and algorithms described in this paper, only the last two were tested. The chance constrained (P1) model is indeed very similar to the model developed for deterministic distance constrained VRPs (Laporte, Desrochers and Nobert [7]). The behaviour of such models is known and well documented (see Laporte and Nobert [10] for deterministic VRPs and Laporte, Louveaux and Mercure [8] for the VRP with stochastic supplies).

In order to assess the performance of models (P2) and (P3), test problems were generated as follows. Distance matrices (c_{ij}) were obtained by first generating points (X_i, X_j) in $[0,100]^2$ according to a uniform distribution, and by then setting $c_{ij} = [(X_i - X_j)^2 + (Y_i - Y_j)^2]$. The number of states varied between 2 and 5. The value of f was equal to zero in all problems. Travel times $t_{ij\ell}$ were equal to $\xi c_{ij}/60$ for all ℓ and for $\xi = 1,2,\ldots|\Xi|$; no service times were considered. The value of B was set equal to 6. If distances are expressed in kilometers and time units in hours, then these parameters translate into travel speeds of $60/\xi$ km/h on all edges, and penalties for trips in excess of 6 hours. The value of γ, the unit penalty cost for excess duration, was allowed to vary between 5 and 20. This again can be compared with an average cost of one unit per kilometer.

Three problems were attempted for n=10, 15, 20 and for each combination of m and the number of states. All tests were run on the Université de Montréal CYBER 855 computer, with a maximum running time of 300 seconds per problem.

Preliminary tests quickly revealed the superiority of (P3) over (P2). The main difficulty with (P2), it seems, lies in the large integer

programs associated with this model, even for relatively modest values of n. Extensive testing of this model was therefore abandoned in favour of (P3).

In order to test the two-index recourse model (P3), we attempted three problems for various combinations of the parameters considered. Problems "failed" when they could not be solved to optimality within 300 seconds or when memory requirements ($250\ 000_8$ words) became exceeded. Failures can be attributed to lack of time or lack of space in approximately equal proportions. Table 1 reports average results over the successful problems. The meanings of the column headings which have not yet been defined are:

STATES : number of stochastic states (i.e. the cardinality of Ξ);

SUCC : number of successful problems out of 3;

PIVOTS : total number of simplex pivots in the branch and bound tree;

CONNEC : total number of type (3) connectivity constraints generated during the branch and bound process;

PENALTY : total number of type (20) penalty constraints generated;

LEVEL : maximum level of the search tree;

TIME : CPU time in seconds (although problems were run on a CYBER 855 computer, all times are CYBER 173 equivalent).

Table 1 : Computational results for the two-index recourse model

r	n	m	STATES	SUCC	PIVOTS	CONNEC	PENALTY	LEVEL	TIME
5	10	2	2	3	23.0	1.0	1.0	0.3	0.23
		2	3	3	51.0	2.3	2.0	2.0	0.70
		2	4	3	54.3	2.0	1.7	4.0	0.88
		2	5	3	64.7	2.0	2.3	2.0	1.23
		3	2	3	23.3	1.0	1.3	0.7	0.24
		3	3	3	28.7	0.7	1.3	1.3	0.35
		3	4	3	57.3	2.0	2.7	3.7	0.88
		3	5	3	45.3	2.0	3.3	2.7	0.96
	15	2	2	2	356.5	3.5	15.5	14.5	11.95
		2	3	3	444.3	3.7	9.7	14.0	16.51
		2	4	3	328.7	3.7	7.7	14.0	12.88
		2	5	3	359.0	5.7	7.3	14.3	15.62
		3	2	1	219.0	4.0	9.0	11.0	6.47
		3	3	2	327.0	3.5	9.5	14.5	11.33
		3	4	3	433.7	3.7	13.0	12.3	16.74
		3	5	3	366.3	3.7	9.3	14.3	16.62
	20	2	2	2	339.0	4.0	9.0	9.5	16.68
		2	3	3	415.7	6.3	8.7	11.7	25.96
		2	4	2	1398.5	3.0	16.0	25.0	103.04
		2	5	1	204.0	2.0	1.0	16.5	14.86
		3	2	2	415.5	2.5	12.0	14.0	19.39
		3	3	3	383.0	2.7	11.7	11.3	22.67
		3	4	2	860.5	7.0	20.5	19.5	61.98
		3	5	1	299.0	2.0	7.0	19.0	27.72
10	10	2	2	3	34.3	1.3	1.3	1.3	0.42
		2	3	3	82.3	2.7	3.7	4.3	1.38
		2	4	3	104.7	2.3	3.7	6.7	1.95
		2	5	3	134.3	2.3	6.3	5.7	3.29
		3	2	3	35.0	1.0	2.3	1.3	0.45
		3	3	3	37.7	1.3	2.3	2.3	0.51
		3	4	3	208.3	2.7	11.0	9.7	4.39
		3	5	3	110.3	2.7	9.0	4.7	2.90
	15	2	2	1	2390.0	4.0	58.0	20.0	98.98
		2	3	2	1381.0	4.0	24.0	20.0	58.16
		2	4	3	1646.0	6.0	32.7	21.0	79.86
		2	5	3	1254.3	7.0	24.3	26.0	64.24
		3	2	0	---	---	---	---	---
		3	3	1	258.0	4.0	10.0	13.0	9.14
		3	4	2	551.5	4.0	19.5	14.5	24.88
		3	5	3	724.0	7.0	17.0	14.5	30.76
	20	2	2	1	750.0	3.0	17.0	16.0	37.22
		2	3	1	862.0	2.0	11.0	22.0	51.22
		2	4	1	1000.0	2.0	11.0	27.0	67.79
		2	5	1	751.0	2.0	11.0	27.0	60.49
		3	2	0	---	---	---	---	---
		3	3	3	2039.3	5.6	37.3	21.7	138.45
		3	4	1	2540.0	4.0	48.0	23.0	185.52
		3	5	1	2446.0	4.0	47.0	27.0	204.05

Table 1 : Computational results for the two-index recourse model (cont'd)

r	n	m	STATES	SUCC	PIVOTS	CONNEC	PENALTY	LEVEL	TIME
15	10	2	2	3	71.3	2.0	3.0	3.3	0.99
		2	3	3	133.3	2.7	5.3	5.7	2.37
		2	4	2	131.5	2.0	5.5	8.0	2.58
		2	5	3	227.0	2.7	8.7	7.0	5.51
		3	2	2	91.0	1.5	10.5	6.0	1.20
		3	3	3	51.7	1.3	2.7	3.0	0.75
		3	4	2	231.5	2.0	10.5	8.5	5.09
		3	5	3	208.3	2.7	15.0	7.0	6.05
	15	2	2	0	---	---	---	---	---
		2	3	1	545.0	6.0	14.0	14.0	18.70
		2	4	1	945.0	6.0	21.0	22.0	37.34
		2	5	2	1401.0	7.0	31.0	23.0	67.51
		3	2	0	---	---	---	---	---
		3	3	1	592.0	4.0	20.0	12.0	19.70
		3	4	0	---	---	---	---	---
		3	5	2	1727.5	7.5	36.5	22.0	76.90
	20	2	2	1	2814.0	4.0	62.0	21.0	179.93
		2	3	1	2943.0	4.0	49.0	25.0	198.35
		2	4	1	3035.0	4.0	43.0	32.0	234.24
		2	5	1	2823.0	4.0	43.0	34.0	249.15
		3	2	0	---	---	---	---	---
		3	3	1	1212.0	6.0	16.0	16.0	73.31
		3	4	0	---	---	---	---	---
		3	5	0	---	---	---	---	---
20	10	2	2	3	107.3	2.3	4.0	4.7	1.59
		2	3	3	200.0	2.7	6.0	8.0	3.49
		2	4	3	334.3	2.3	10.0	10.0	6.75
		2	5	3	290.7	3.0	13.0	8.0	7.50
		3	2	2	21.0	1.5	0.5	0.0	0.19
		3	3	3	76.0	1.7	3.3	5.3	1.15
		3	4	2	338.0	2.0	14.0	10.0	7.85
		3	5	3	379.7	3.3	21.7	7.0	11.82
	15	2	2	0	---	---	---	---	---
		2	3	1	1359.0	6.0	30.0	17.0	49.91
		2	4	1	2253.0	6.0	41.0	25.0	98.56
		2	5	2	2971.0	7.0	47.0	31.5	153.69
		3	2	0	---	---	---	---	---
		3	3	1	932.0	5.0	35.0	20.0	35.67
		3	4	0	---	---	---	---	---
		3	5	0	---	---	---	---	---

Our results indicate that problem difficulty increases with γ. This was to be expected since a large value of γ requires the generation of more penalty constraints, which in turns leads to more pivots and larger CPU times. At the other exteme, setting γ at a value close to 0 would have made the problem almost as easy as an m-TSP. Problem difficulty is slightly affected by the number of states which determines the number of type (25) constraints. The number of vehicles used, on the other hand, seems to have no effect on the performance of the algorithm. Problems involving a large number of cities are of course harder to solve, mainly because they involve more integer variables and require more branching; this is reflected by the values of the LEVEL column. The small number of connectivity and penalty constraints generated in most problems indicates that our relaxation approach was quite effective.

7. CONCLUSION

We have modeled and solved in this paper a difficult class of stochastic VRPs arising in a number of practical situations. The chance constrained model is similar to those developed for deterministic VRPs. Two recourse models were presented: one uses three-index variables and explicitly lists all penalty constraints; the second recourse model contains two-index variables and can be solved by means of a constraint relaxation algorithm. Computational results indicate that it can be used for the exact solution of moderate size problems.

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